Some aspects of the translog production function estimation

*Author**:

Florin-Marius PAVELESCU

A*bstract.* *In a translog production function, the number of parameters practically ”explodes” as the number of considered production factors increases. Consequently, the shortcoming in the estimation of the respective production function is the occurrence of collinearity. Theoretically, the collinearity impact is minimum if a single production factor is taken into account. In this case, we can determine not only the output elasticity but also the elasticity of scale related to the respective production factor. In the present paper, we demonstrate that the relationship between the output elasticity and estimated average elasticity of scale depends on the dynamics trajectory of the production factor, underexponential and overexponential, respectively. At the end, a practical example is offered, dealing with the computation of the Gross Domestic Product elasticity and average elasticity of scale related to employed population in the United Kingdom and France during 1999-2009.*

Keywords: *estimation constraints, informational energy, translog multiplier, augmented output elasticity related to a production factor.*

JEL Clasification: C13, C20, C51, C52

1. A short history of translog production function

The translog production functions occurred in the context of researches related to the discovery and definition of new flexible forms of production functions and to the approximation of CES production function. In fact, the first form of a

* Florin-Marius PAVELESCU Ph. D., Scientific research I, Institute of National Economy, Romanian Academy, e-mail: pavelescu.florin@yahoo.com.

translog production may be considered the proposal made in 1967 by J. Kmenta for the approximation of the CES production function with a second order Taylor series, when the elasticity of substitution is very close to the unitary value, which is the case of Cobb-Douglas production function. The form of the above-mentioned production function is:

$$\ln Y = \ln A_3 + \alpha_3 \cdot \ln K + \beta_3 \cdot \ln L + \chi_3 \cdot \ln^2 (K/L) \tag{1}$$

where:

ln= natural logarithm

Y= Output (Gross domestic product)

K= Fixed capital

L= Employed population.

A_3, α_3, β_3, γ_3 are parametres to be estimated.

In 1971, Grilichs and Ringstad proposed new forms of production function. The first one was obtained by imposing the condition that $\alpha+\beta=1$. This way, the production function became in fact a labour productivity function:

$$\ln(Y/L) = \ln A_2 + \alpha_2 \cdot \ln(K/L) + \chi_2 \cdot \ln^2 (K/L) \tag{2}$$

It is to be noticed that the above-mentioned function is one of a second order polynomial in the logarithms of the single input considered, capital-labour ratio, respectively.

The second form of production function was defined in conditions of relaxing the constraints imposed to the parameters in the Kmenta function, in order to test the homotheticity assumptions, and was written as:

$$\ln Y = \ln A_{KL} + \alpha_K \cdot \ln K + \alpha_L \cdot \ln L + \beta_{K^2} \ln^2 K + \beta_{L^2} \cdot \ln^2 L + \beta_{KL} \cdot \ln K \cdot \ln L \tag{3}$$

In fact, the same production function was used by Sargant also in 1971 and was called a log-quadratic one. It is important to mention that the term “translog production function”, abridged from “ transcendental logarithmic production function “was proposed by Christiansen, Jorgensn and Lau in two papers published in 1971 and 1973, which dealt with the problems of strong separability (additivity) and homogeneity of Cobb-Douglas and CES production functions and their implications for the production frontier. The generalised form of translog production function, which takes into account a number of n inputs (production factors), can be exprssed as:

$$\ln Y = \ln A_{\alpha_i,\beta_{ij}} + \sum_{i=1}^{n} \alpha_i \cdot \ln X_i + (\frac{1}{2}) \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} \cdot \ln X_i \cdot \ln X_j \qquad (4)$$

The translog production functions represent in fact a class of flexible functional forms for the production functions (Ch. Allen, St. Hall, 1997). One of the main advantages of the respective production function is that, unlike in case of Cobb-Douglas production function, it does not assume rigid premises such as: perfect or "smooth" substitution between production factors or perfect competition on the production factors market (J.Klacek, et al., 2007). Also, the concept of the translog production function permits to pass from a linear relationship between the output and the production factors, which are taken into account, to a non-linear one. Due to its properties, the translog production function can be used for the second order approximation of a linear-homogenous production, the estimation of the Allen elasticities of substitution, the estimation of the production frontier or the measurement of the total factor productivity dynamics.

2. Main indicators and constraints in the estimation of translog production functions parameters

In a translog production function, the marginal product ($\frac{\partial Y}{\partial X_i}$) is equal to:

$$\frac{\partial Y}{\partial X_i} = \alpha_i + \sum_{j=1}^{n} \beta_{ij} \cdot \ln X_j \qquad (5)$$

It is to be noted that the marginal product of a translog production function is formally a Cobb-Douglas production function.

Having in view the marginal product, we can also determine the marginal rate of transformation between two production factors ($\frac{\partial X_j}{\partial X_i}$):

$$\frac{\partial X_j}{\partial X_i} = \frac{\alpha_i + \sum_{i=1}^{n} \beta_{ij} \cdot \ln X_i}{\alpha_j + \sum_{j=1}^{n} \beta_{ij} \cdot \ln X_j} \qquad (6)$$

It is important to mention that C.E. Ferguson (1979) demonstrated that **the marginal product is equal to the elasticity of scale.**

As we have shown before, the translog production function has potentially a series of advantages in the research of the economic activity from the theoretical point of view. But the great number of parameters that have to be estimated in order to make operational the concept of "translog production function" impose hard constraints on the result feasibility, because the occurrence of an extended collinearity is favoured. In fact, the number of the parameters practically "explodes" as the number of production factors, which are taken into account increase. If the number of production factors is equal to n, the number of estimated parameters is equal to $\frac{n \cdot (n+3)}{2}$.

If the OLS method is used in estimation, even if only three production factors are considered, the probability of the occurrence of the harmful collinearity is very high[1].

A solution used in order to surpass the difficulties generated by the collinearity is the ridge regression, which theoretically permits to obtain estimations which are not distorted by high degree of collinearity and especially by the harmful one. But the ridge regression has also a shortcoming, i.e. the ridge (correction) parameter used in order to diminish the collinearity impact is in fact subjectively chosen (J. Klacek, J. Vopravil, 2008). Therefore, when the ridge regression is used, we are not sure whether the solution obtained is an optimal one. Also, the deviation of the results obtained in the context of ridge regression tends to be greater and greater as the number of production factors is higher and higher.

Consequently, another solution in order to obtain feasible results with translog production is the limitation of the number of production factors to those which can be really considered for the behaviour of the output. Because the collinearity is a cumulative phenomenon, the first test for introduction in the translog production function of a specific production factor is to estimate, by the OLS method, of the translog production function related only to the analysed factor. In case that the results obtained in estimation are considered feasible, the

[1] The collinearity occurs in any estimation of an econometric model, being determinated by the correlations between explanatory variables. The collinearity is seen as "harmful" if the sign of at least one estimated parameter is contrary to the sign of the coefficient of correlation between the resultative variable and the analysed explanatory variable. For more details, see F. M. Pavelescu, 2010b.

respective production factor may be introduced into an extended translog production function.

3. Characteristic features of the estimated parameters of the translog production function with a single production factor

If the OLS method is used in order to estimate the parameters of a translog production function with a single production factor, respectively,

$$\ln Y = \ln A_2 + \alpha_{2X} \cdot \ln X + (\frac{1}{2}) \cdot \beta_{2X} \cdot \ln^2 X \quad (7)$$

the estimated parameters can be written:

$$\ln A_2 = (\ln Y)_{med} - \alpha_k \cdot (\ln X)_{med} - (\frac{1}{2}) \cdot \beta_k \cdot (\ln^2 X)_{med} \quad (8)$$

$$\alpha_{2k} = \alpha_{1k} \cdot T_{\ln X} \quad (9)$$

$$\beta_{2X} = \beta_{1X} \cdot T_{\ln^2 X} \quad (10)$$

where:

$(\ln Y)_{med}$ = arithmetical mean of the natural logarithms of the output indices

$(\ln X)_{med}$ = arithmetical mean of the natural logarithms of the production factor indices

$(\ln^2 X)_{med}$ = arithmetical mean of squares of the natural logarithms of the production factors indices

$$\alpha_{1X} = \frac{\operatorname{cov}(\ln Y; \ln X)}{D^2(\ln X)} \quad (11),$$

$$\beta_{1X^2} = \frac{2 \cdot \operatorname{cov}(\ln Y; \ln^2 X)}{D^2(\ln^2 X)} \quad (12)$$

and represent the proper estimated values of parameter α and β in case of unifactorial linear regressions:

$$\ln Y = \ln A_{1X} + \alpha_1 \cdot \ln X \quad (13)$$

and

$$\ln Y = \ln A_{1X^2} + (\frac{1}{2}) \cdot \beta_1 \cdot \ln^2 X \tag{14}$$

Cov (lnY;lnX) = covariance between natural logarithms of output indices and natural logarithms of production factor indices.

D^2 (lnX) = variance of natural logarithms of production factor indices.

Cov (lnY;$\ln^2$ X) = covariance between natural logarithms of output indices and the square of natural logarithms of production factor indices.

D^2 ($\ln^2$ X) = variance of the square of natural logarithms of production factor indices.

$T_{\ln X}$ and $T_{\ln^2 X}$ represent the coefficients of alignment to collinearity hazard related to explanatory variables lnX and $\ln^2$X:

$$T_{\ln X} = \frac{1 - R(\ln X; \ln^2 X) \cdot r}{1 - R^2(\ln X; \ln^2 X)} \tag{15}$$

$$T_{\ln^2 X} = \frac{r - R(\ln X; \ln^2 X)}{r \cdot (1 - R^2(\ln X; \ln^2 X))} \tag{16}$$

$$r = \frac{R(\ln Y; \ln^2 X)}{R(\ln Y; \ln X)} \tag{17}$$

R(lnY;$\ln^2$X) = Pearson coefficient of correlation between the natural logarithm of output indices and the square of natural logarithm of production factor.

R (lnY; lnX) = Pearson coefficient of correlation between the natural logarithm of output indices and the natural logarithm of production factor.

It can be observed that the absolute values of ratio r plays an essential role in modelling the values of the coefficients of alignment to collinearity hazard[1] in case of a linear regression with two explanatory variables. In a translog production function, r and R(lnX ;$\ln^2$X) usualy have the same sign.

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[1] It is to be mentioned that ratio r was defined in F. M. Pavelescu, 2010b as *the m.r.v. (mediated by resultative variable) coefficient of correlation between the explanatory variables.* The respective indicator is computed only related to the Pearson coefficient of correlation with the highest absolute value.

Therefore, if /r/<1, and we obtain $T_{\ln X} > T_{\ln^2 X}$ and consequently lnX acts as the main explanatory variable and $\ln^2 X$ as the secondary explanatory variable.

If /r/>1, and we obtain $T_{\ln X} < T_{\ln^2 X}$ and therefore we may consider lnX as the secondary explanatory variable and $\ln^2 X$ as the main explanatory variable.

Having in mind the computation formulae mentioned above, we may conclude that the coefficients of alignment to collinearity hazard determines that the estimated parameters of a multiple regression to be the derived ones in comparison with the proper values obtained in case of the simple regressions related to the analysed production factor (F.M. Pavelescu, 2005)[1]. Also, if we have in mind the concepts of „signal" and „noise", used in (Belsey, 1991), we may define parameters estimated in simple linear regressions as the „initial signal" and the parameters estimated in a multiple regression as the „signal distorted by noise". Therefore, coefficients of alignment to collinearity hazard represents in fact the „signal to noise ratio".

The values of the coefficient of alignment to collinearity hazard represent also a premise for the identification and classification of the collinearity type. In F.M. Pavelescu, 2010b, the following classification of the collinearity thet may occur in a multiple regression was proposed:

a) **Weak collinearity**, if all the coefficients of alignment to collinearity are at least equal to 0.5.

b) **Degrading collinearity** if all the coefficients of alignment to collinearity hazard are positive and at least one of them is smaller than 0.5.

c) **Harmful collinearity** if at least one of the coefficients of alignment to collinearity hazard is negative.

In case of a linear regression with two explanatory variables, the relationship between the absolute values of coefficient r and R(lnX; $\ln^2$X) are essential for the classification of collinearity.

Therefore, **weak collinearity may be considered if:**

[1] It is important to notice that the coefficients of alignment to collinearity hazard influence not only the estimated values of the parameters of a multiple regression, but also the computed values of some statistical tests such as: coefficient of partial correlation,coefficient of determination, adjusted coefficient of determination, Fisher test, Student Test (see F.M. Pavelescu, 2009, 2010 a).

$$\frac{2 \cdot // R(\ln X ; \ln^2 X) //}{1+R^2(\ln X ; \ln^2 X)} < // r // < \frac{1+R^2(\ln X ; \ln^2 X)}{2 \cdot // R(\ln X ; \ln^2 X) //} \quad (18)$$

Degrading collinearity is identified in two situations, if:

$$// R(\ln X ; \ln^2 X) // < // r // < \frac{2 \cdot // R(\ln X ; \ln^2 X) //}{1+R^2(\ln X ; \ln^2 X)} \quad (19)$$

or

$$\frac{1+R^2(\ln X ; \ln^2 X)}{2 \cdot // R(\ln X ; \ln^2 X) //} < // r // < \frac{1}{// R(\ln X ; \ln^2 X) //} \quad (20)$$

Harmful collinearity occurs also in two situations, if:

$$// r // < // R(\ln X ; \ln^2 X) // \quad (21)$$

or

$$// r // > \frac{1}{// R(\ln X ; \ln^2 X) //} \quad (22)$$

It is to be mentioned that the occurrence of harmful collinearity makes the estimation results to be completely unfeasible. The respective form of collinearity may be easily identified by the Corrected Student Test, which permits to detect the sign of the coefficient of alignment to collinearity hazard (see details in F.M. Pavelescu, 2009). The presence of degrading collinearity may be identified as a "classical form" of collinearity in conditions of low and very low computed values of the (Standard) Student Test.

4. Correlation between the estimated values of output elasticity and average elasticity of scale in case of a translog production function with a single production factor

As we have previously mentioned, one of the indicators that can be defined and computed in conditions of a translog production function is the elasticity of scale. Having in mind the formula (5), it is possible to compute the elasticity of scale for quantity allocated from production factors in each year of the analyzed period. The respective indicator may also be computed as the average for the entire period. This way, we can offer synthetic information related to the correlation between the scale of the allocated quantity from the production factors and their efficiency, measured by the estimated values of the elasticity of output related to the production factors.

In case of a translog production function with a single production factor, the estimated average elasticity of scale (E_{smed}) may be written:

$$E_{smed} = \alpha_{1X} \cdot \left(\frac{1 - r \cdot R(\ln X; \ln^2 X)}{1 - R^2(\ln X; \ln^2 X)} + 2\ln X_R \cdot \frac{D(\ln X)}{D(\ln^2 X)} \cdot \frac{r - R(\ln X; \ln^2 X)}{1 - R^2(\ln X; \ln^2 X)}\right) \quad (23)$$

where:

$\ln X_R$= natural logarithm of representative index of production factor X[1].

If we write:

$$M_{TrX} = \frac{1 - r \cdot R(\ln X; \ln^2 X)}{1 - R^2(\ln X; \ln^2 X)} + 2\ln X_R \cdot \frac{D(\ln X)}{D(\ln^2 X)} \cdot \frac{r - R(\ln X; \ln^2 X)}{1 - R^2(\ln X; \ln^2 X)} \quad (24)$$

we may conclude that the estimated elasticity of scale is the product between the estimated proper elasticity related to the analyzed production factor (α_{1X}) and estimated translog multiplier (M_{TrX}). Consequently, **the estimated elasticity of scale in case of a translog production function with a single production factor represents in fact an augmented elasticity of output related to the analyzed production factor**.

The computation formulae mentioned above reveal the fact that the estimated elasticity of scale is the result of the impact of many modeling factors. In these conditions it is very useful to analyse the modeling factors of the output elasticity, on the one hand, and of the translog multiplier, on the other hand.

The **values of estimated proper elasticity of output** related to the analyzed production factor have to be interpreted in correlation with values of the logarithm of the representative index of the output ($\ln Y_R$) and the correlation between the dynamics of quantity allocated from the production factor and the dynamics of the respective factor productivity.

It is very important to have in mind that:

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[1] Representative index was defined in F.M. Pavelescu, 1986 as the geometrical mean of the indices with fixed basis.

$$\alpha_{1X} = 1 + \frac{D(\ln(Y/X))}{D(\ln X)} \cdot R(\ln(Y/X); \ln X) \qquad (25)$$

where:

D(ln X), D(ln (Y/X)) = standard deviation of the logarithm of indices of allocated quantities from the analyzed production factor, on the one hand, and of productivity of the analyzed factor, on the other hand.

R(ln(Y/X);lnX) = Pearson coefficient of correlation between indices of productivity and allocated quantities from the analyzed production factor.

Having in view the analytical premises mentioned above, we may find six types of output dynamics related to the dynamics of quantity allocated from the analyzed production factor and the respective production factor productivity (Table 1).

Table 1. Correlations between the logarithm of output representative index, output estimated proper elasticity and production factor allocated quantity and productivity

$\ln Y_R$	α_{1X}	Significance
$LnY_R>0$	$\alpha_{1X}>1$	Increase in the output in conditions of simultaneous increase in quantity allocated and productivity of the analyzed production factor
$LnY_R>0$	$0<\alpha_{1X}<1$	Increase in the output level in conditions of increase in quantity allocated and decrease in productivity of the analyzed production factor
$LnY_R>0$	$\alpha_{1X}<0$	Increase in the output level in conditions of decrease in quantity allocated and increase in productivity of the analyzed production factor
$LnY_R<0$	$\alpha_{1X}>1$	Decrease in the output level in conditions of simultaneous decrease in quantity allocated and productivity of the analyzed production factor
$LnY_R<0$	$0<\alpha_{1X}<1$	Decrease in the output level in conditions of a decrease in quantity allocated and increase in productivity of the analyzed production factor
$LnY_R<0$	$\alpha_{1X}<0$	Decrease in the output level in conditions of increase in quantity allocated and decrease in productivity of the analyzed production factor

The values of the estimated elasticity of the output related to the analyzed production factor (α_{1X}) may be written as a product of three variables, respectively:

$$\alpha_{1X} = \frac{\ln Y_R}{\ln X_R} \cdot \frac{Cv(\ln Y)}{Cv(\ln X)} \cdot R(\ln Y; \ln X) \tag{26}$$

where:

$\ln X_R$= logarithm of the representative index of the explanatory variable X.

Cv(ln Y), Cv(lnX) = coefficient of variation of the logarithm of indices of the resultative variable and the explanatory variable , respectively.

This way, it can be revealed that an important impact on the value of the output elasticity related to the analyzed production factor has the ratio of the logarithm of the representative indices of the resultative the explanatory variable.

Another modeling factor of the estimated output elasticity is the ratio $\frac{Cv(\ln Y)}{Cv(\ln X)}$, which measures the characteristic feature of the distribution in time of the resultative variable relative to the distribution in time of the explanatory variable.

The third modeling factor of the output elasticity related to the analyzed production factor is the Pearson coefficient of correlation R(lnY;lnX), which determines the degree of functionality of the relationship between the resultative and explanatory variable.

It is important to observe that if R(lnY;lnX)=1, also the product $\frac{Cv(\ln Y)}{Cv(\ln X)} \cdot R(\ln Y; \ln X) = 1$. In other words, if the ln Y and lnX are in a functional relationship, the estimated output elasticity related to a production factor is the ratio of the logarithm of representive index of resultative variable to the representative index of explanatory variable.

Related to **the analysis of the modeling factors of translog multiplier** we first write:

$$s = (\ln X)_{med} \cdot \frac{D(\ln X)}{D(\ln^2 X)} \tag{27}$$

Consequently, the respective indicator may be expressed as:

$$M_{TrX} = \frac{1 - 2s \cdot R(\ln X; \ln^2 X) + r \cdot (2s - R(\ln X; \ln^2 X))}{1 - R^2(\ln X; \ln^2 X)} \quad (28)$$

The ratio $\frac{2s}{R(\ln X; \ln X^2)}$, may be also written

$$\frac{2s}{R(\ln X; \ln X^2)} = \frac{2}{n} \cdot \frac{\sum_{i=1}^{n} \ln X_i \cdot (n \cdot \sum_{i=1}^{n} \ln^2 X - (\sum_{i=1}^{n} \ln X)^2}{n \cdot \sum_{i=1}^{n} \ln^3 X - \sum_{i=1}^{n} \ln X \cdot \sum_{i=1}^{n} \ln^2 X} \quad (29)$$

If we write:

$$g_i = \frac{\ln X_i}{\sum_{i=1}^{n} \ln X_i} \quad (30)$$

we obtain:

$$\frac{2s}{R(\ln X; \ln X^2)} = \frac{2}{n} \cdot \frac{n \cdot \sum_{i=1}^{n} g_i^2 - 1}{n \cdot \sum_{i=1}^{n} g_i^3 - \sum_{i=1}^{n} g_i^2} \quad (31)$$

If all $\ln X_i$ are positive, we may say that the relationship between s and $R(\ln X;\ln^2 X)$ is modeled to a great extent by the informational energy of $\ln X_i$, ($\sum_{i=1}^{n} g_i^2$) or, in other words, by the distribution in time series of the above - mentioned explanatory variable.

Having in view the formula (31) it is easier to determine the values of the ratio $\frac{2s}{R(\ln X; \ln X^2)}$ in particular cases, as follows:

1) If all the values of $\ln X_i$ are practically equal to $\ln X_R$,, the ratio $\frac{2s}{R(\ln X; \ln X^2)}$ tends to 2.

2) If the dynamics of the production factor X is strictly exponential, we obtain $\frac{2s}{R(\ln X;\ln X^2)}=1$

3) If the values $\ln X_i$, tend to be very concentrated in a point and determine that $\sum_{i=1}^{n} g_i^2$ and $\sum_{i=1}^{n} g_i^3$ tend to 1 the ratio $\frac{2s}{R(\ln X;\ln X^2)}$ tends to $\frac{2}{n}$.

Having in mind that $\ln X_i$, represent time series, we may admit that conventionally the ratio $\frac{2s}{R(\ln X;\ln X^2)}$ reveals the feature of the dynamics trajectory of the production factor X.

If $1<\frac{2s}{R(\ln X;\ln X^2)}<2$**, the dynamics trajectory of the production factor X is conventionally underexponential.**

If $\frac{2}{n}<\frac{2s}{R(\ln X;\ln X^2)}<1$**, the dynamics trajectory of the production factor X is conventionally overexponential.**

If we take into account as feasible estimations only the situations when the weak and degrading collinearity occurs, we find that $M_{TrX}=1$, if $r=R(\ln x;\ln^2 X)$ and $M_{TrX}=\frac{2s}{R(\ln X;\ln^2 x)}$ if $r=\frac{1}{R(\ln X;\ln^2 X)}$.

In this context, we may conclude that:

a) **the estimated average elasticity of scale is greater than the output elasticity if the dynamics trajectory of the production factor is conventionally underexponential,** and

b) **the estimated average elasticity of scale is smaller than the output elasticity if the dynamics trajectory of the production factor is conventionally overexponential.**

5. A factorial analysis model of the translog multiplier

The feature of the dynamics trajectory of the production factor has an important impact on the values of the translog multiplier, but it is not the only modeling factor. The other modeling factor is the values taken by the rartios r. Therefore, we can build a factorial analysis model of the translog multiplier.

The above-mentioned factorial analysis model also implies the computation of two additional indicators.

The first additional indicator is the computation of the translog multiplier when all the values of $\ln X_i$ are practically equal to $\ln X_R$. Under these conditions, the above-mentioned indicator tends to 1.5. Having in view this result, we may consider that the reference value of the translog multiplier (M_{TrXref}) is equal to 1.5.

The second indicator is related to the special case when r=1. In other words, we are, theoretically, in a situation when the values of the logarithms of indices of the explanatory variable X are differentiated in their evolution in time, but the correlation of lnX and $\ln^2 X$, respectively, with lnY have the same intensity. In this situation, the computed value of the translog multiplier ($M_{TrXdintraject}$) is:

$$M_{TrXd\,\mathrm{int}\,raject} = \frac{1+2s}{1+R(\ln X;\ln^2 X)} \tag{32}$$

Based on the indicators M_{TrXref} and $M_{TrXdintraject}$ we can identify the influences of the dynamics trajectory of the analyzed production factor ($\Delta d\,\mathrm{int}\,raject$) and of the differentiation in intensity of Pearson correlation between ln Y and lnX, on the one hand, and between the lnY and $\ln^2 X$, on the other hand ($\Delta difr$).

A methodological problem to be solved when the factorial analysis model is implemented refers to the situation when $R(\ln X:\ln^2 X)$ is negative. Usually, in this context, r and s are also negative. Therefore, the sign of these modeling factors does not practically influence the estimated translog multiplier. In order to avoid the problems generated by the negative sign of the above-mentioned modelling factors, in computing the indicators M_{TrXref} and $M_{TrXdintraject}$, we will use the absolute values of $R(\ln X:\ln^2 X)$, r and s.

The influence of the dynamics trajectory of the production factor ($\Delta d\,\mathrm{int}\,raject$) is:

$$\Delta d\,\mathrm{int}\,raject = 1.5 - \frac{1+2\,//\,s\,//}{1+//\,R(\ln X;\ln^2 X)\,//} \tag{33}$$

It is important to notice that $\Delta d\,\mathrm{int}\,raject$ <0.

$$\Delta difr = \frac{1-2s\cdot R(\ln X;\ln^2 X)+r\cdot(2s-R(\ln X;\ln^2 X)}{1-R^2(\ln X;\ln^2 X)} - \frac{1+2//s//}{1+//R(\ln X;\ln^2 X)//} \quad (34)$$

The sign of $\Delta difr$ depends both on the dynamics trajectory of the production factor and on the role of the explanatory variable lnX in the translog production function.

Therefore, we may find four situations, as follows:

a) $\Delta difr$ <0, if the dynamics trajectory is underexponential and lnX is the main explanatory variable.

b) $\Delta difr$ >0, if the dynamics trajectory is underexponential and lnX is the secondary explanatory variable.

a) $\Delta difr$ >0, if the dynamics trajectory is overexponential and lnX is the main explanatory variable.

b) $\Delta difr$ <0, if the dynamics trajectory is overexponential and lnX is the secondary explanatory variable.

6. Two numerical examples. The translog production function related to employed population in the United Kingdom and France during 1998 and 2009

We will further investigate the modeling factors of the correlation between the output elasticity and the average elasticiticity of scale related to an analyzed production input (factor), having in view the dynamics of Gross Domestic Product (Y) and employed population (L) in the United Kingdom and France during 1999 and 2009, based on data presented in “Employment in Europe 2010” (Statistical Annex) Brussels, November 2010.

The estimations of the simple linear regression between the lnY and lnL give the following results:

For the United Kingdom $\ln Y=0.0042+2.6582\cdot \ln L$ $R^2=0.9898$
(0.7279) (29.5052)

For France $\ln Y=-0.0248+2.1014\cdot \ln L$ $R^2=0.9335$
(-1.7014) (11.2421)

R^2= the coefficient of determination. In brackets, we present the computed value of the standard Student Test.

The estimated proper elasticity of Gross Domestic Product (output) and the employed population (production factor), respectively, α_{1L} is bigger than 1 in both cases and is obtained in conditions of increase in the Gross Domestic Product (Table 2).

Table 2. The modeling factors of the proper elasticity of Gross Domestic Product related to employed population in the United Kingdom and France during 1999 and 2009 (1998=100)

Indicator	United Kingdom	France
A_{1L}	2.6582	2.1014
LnY_R	0.1605	0.1312
LnL_R	0.0588	0.0743
$\frac{\ln Y_R}{\ln L_R}$	2.7250	1.7668
$\frac{\ln Cv(\ln Y)}{\ln Cv(\ln L)}$	0.9881	1.1939
R(lnY;lnX)	0.9949	0.9662

Therefore, we may identify a positive correlation between the dynamics of the employed population and labour productivity growth at the whole economy level of the United Kingdom and France. It is to be noted that the ratio $\frac{\ln Y_R}{\ln L_R}$ is sensibly greater for the United Kingdom (2.7250) in comparison with France (1.7668).

The product $\frac{\ln Cv(\ln Y)}{\ln Cv(\ln L)} R \cdot (\ln Y; \ln L)$ is equal to 0.9830 for the United Kingdom and 1.1535 for France. The respective results explain the values estimated for the parameter A_{1L}, respectively positive, but very near to 0 for the United Kingdom and negative for France.

The determination of the average elasticity of scale of the Gross Domestic Product related to the employed population implies the computation of the other two regressions, as follows:

$\ln Y = \ln A_{1Ltr} + \beta_{1Ltr} \cdot \ln^2 L$ and $\ln Y = \ln A_{2L} + \alpha_{2L} \cdot \ln L + \beta_{2Ltr} \cdot \ln^2 L$.

The estimation results are:

For the United Kingdom

$\ln Y = 0.0659 + 22.9033 \cdot \ln^2 L$ $\quad R^2 = 0.9406$
$\quad (6.7556) \quad (11.9414)$

$\ln Y = -0.0065 + 3.15568 \cdot \ln L - 4.4831 \cdot \ln^2 L$
$\quad (-0.5681) \quad (6.7221) \quad (-1.080)$

Consequently, the average elasticity of scale of the Gross Domestic Product related to employed population (E_{smed}) is equal to 2.6286

For France

$\ln Y = 0.0358 + 22.9033 \cdot \ln^2 L$ $\quad R^2 = 0.9387$
$\quad (3.8812) \quad (11.7351)$

$\ln Y = 0.0071 + 0.9515 \cdot \ln L + 8.7513 \cdot \ln^2 L$ $\quad R^2 = 0.9737$
$\quad (0.2824) \quad (1.2150) \quad (1.5064)$

Consequently, the average elasticity of scale of the Gross Domestic Product related to employed population (E_{smed}) is equal to 2.2514.

At first sight, the average elasticity of scale of the Gross Domestic Product related to employed population is comparable for the United Kingdom and France. A careful analysis of the modelling factors of the above-mentioned indicator shows important differences between the two estimates.

Therefore, the ratio $\dfrac{D(\ln L)}{D(\ln^2 L)}$ is 8.8382 for the United Kingdom and 7.4186 for France (Table 3). The ratio $r = \dfrac{R(\ln Y; \ln^2 L)}{R(\ln Y; \ln L)}$ is 0.9748 for the United Kingdom and 1.0027 in case of France. On this basis, we may conclude that lnL is the main explanatory variable in the translog production function estimated for the United Kingdom and secondary explanatory variable in the above-mentioned function estimated for France.

Since for the United Kingdom $r < R(\ln L; \ln^2 L)$ we are faced with a harmful collinearity, illustrated by the fact that $T_{\ln L} = 1.1873$ and $T_{\ln^2 L} = -0.1957$ (table 3).

Table 3. The modeling factors of the estimated average elasticity of scale related to employed population in the United Kingdom and France during 1999 and 2009 (1998=100)

Indicator	United Kingdom	France
$\frac{D(\ln L)}{D(\ln^2 L)}$	8.8382	7.4186
2s	1.0402	1.0195
R(lnl;ln2L)	0.9818	0.9747
$\frac{1}{R(\ln L;\ln^2 L)}$	1.0186	1.0266
$\frac{2R(\ln L;\ln^2 L)}{1+R^2(\ln L;\ln^2 L)}$	0.9998	0.9996
$\frac{1+R^2(\ln L;\ln^2 L)}{2R(\ln L;\ln^2 L)}$	1.0002	1.0003
r	0.9748	1.0027
$T_{\ln L}$	1.1873	0.4528
$T_{\ln^2 L}$	-0.1957	0.5598

For France both coefficients of alignment to collinearity hazard are positive, $T_{\ln L} = 0.4582$ and $T_{\ln^2 L} = 0.5594$. The collinearity is a degrading one, because $r > \frac{1+R^2(\ln L;\ln^2 L)}{2R(\ln L;\ln^2 L)}$. Also, it is to note that the above-mentioned feature of collinearity determines low computed values of the (Standard) Student Test related to explanatory variables lnL (1.2150) and $\ln^2$L (1.5064).

In both examples, because 2s>R(lnL;$\ln^2$L), the dynamics of employed population is an underexponential one. In this context, the translog multiplier (M_{TrL}) is 0.9889 for the United Kingdom and 1.0714 the France (Table 4). This way, the theoretical assumptions related to the interdependence of the dynamics trajectory and the collinearity features are validated.

Table 4. The factorial analysis of the translog multiplier related to employed population in the United Kingdom and France during 1999 and 2009 (1998=100)

Indicator	United Kingdom	France
M_{TrLref}	1.5000	1.5000
$M_{TrLdintraject}$	1.0295	1.0644
M_{TrL}	0.9889	1.0714
$\Delta d\,\mathrm{int}\,raject$	-0.4705	-0.4356
$\Delta difr$	-0.0406	0.0070

N.B. M_{TrLref}, $M_{TrLdintraject}$ and M_{TrL} have similar significance to M_{TrXref}, $M_{TrXdintraject}$ and M_{TrX}.

The factorial analysis shows that the dynamics trajectory of the employed population plays the main role in the decrease of the estimated translog multiplier from its reference value by -0.4705 in the United Kingdom and by -0.4356 in France. The impact of ratio r on the value of translog multiplier is differentiated in the two examples. The respective impact is negative and quite important in absolute value for United Kingdom, while for the France it is positive, but with a very small absolute value.

Conclusions

The estimation of the parameters of a translog production with a single production factor permits to enlarge the vision related to the relationship between the output and the analyzed production factor from a linear one to a non-linear one. In this context, we can emphasize the role of the acceleration of the dynamics of the production factors in order to increase the output level. In the context of weak and degrading collinearity, an accelerated dynamics (overexponential) of the production factor determines a translog multiplier smaller than 1, while a less accelerated dynamics (underexponential) determines a translog multiplier bigger than 1. In other words, the elasticity of scale tends to decrease in comparison with the output elasticity as the dynamics of the production factor become more and more non-linear.

Even in conditions of a single production factor, the problem of collinearity in the translog production function is not eliminated. The values of $R(\ln X;\ln^2 X)$ are quite high, so the incidence of harmful collinearity may be considerable. On the other hand, if the harmful collinearity does not occur, the most frequent feature of the respective phenomenon is the degrading case.

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