Features of the production factors substitution and the estimated parameters of the Cobb-Douglas production function with constant returns to scale and disembodied technical change

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Abstract. *The paper shows that, if the OLS method is used, the features of substitution between the considered production factors have an important impact on the estimated parameters of the Cobb-Douglas production function with constant returns to scale and disembodied technical change as well as on the dynamics of the total factor productivity. The author reveals the dual nature of the above-mentioned production function and identifies all the possible correlations of the estimated parameters when productivity of labour tends to grow in the long run. A special attention is paid to the possible occurence of harmful collinearity. The author reveals that the occurence of the respective type of collinearity type is dependent on the feature of the dynamics of productivity of the substituting production factor. Consequently, an extended estimation methodology is proposed in order to reveal the feature of the production factors substitution, the form of the trajectory o f thedynamics of the partial productivity of the considered production factors and the impact of colinearity on the estimated parameters. This methodology is practically used by considering the evolution of Japan`s economy during the 1971-1986 period.*

Keywords*: productivity function, representative index, total factor productivity, capital deeping, substituting production factor, harmful collinearity,*

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1. Introduction

The production function is one of the key-concepts of neo-classical economy. The forms and variants of the production functions are quite impressive. Among the most used production functions in theoretical and practical investigations of the economic processes we find the Cobb-Douglas production function with constant returns and Cobb-Douglas production with constant returns and disembodied technical change. The practical use of these production functions implies the estimation of their parameters. But, in many cases, the results obtained are quite hazardous and conequently they are difficult to be correctly interpretated. These situations are caused both by the relaxation of the initial assumptions related to the significance of the production function parameters and by the algebraical properties of the estimation method. Hence, a careful investigation of the theoretical assumption related to the parameters of the Cobb-Douglas production function and of the impact of the algebraical properties of the estimation method can really improve the interpretation of the dynamics of output and of the considere production factors productivity during the analysed period.

2. The initial form of the Cobb-Douglas production function

The concept of the Cobb-Douglas production function was proposed in 1928, in the context of the analysis of the statistical data, which showed that, during the 1908-1918 period, in the USA economy, the weight of the wages in the total gross value added remained practically constant, at about 74%, in spite of the sensible change of the capital deeping *i.e*. fixed capital /employment ratio (K. C. Border, 2004).

Considering the assumptions of a constant weight of the labour force remuneration in the gross value added, and of the profit-maximization, a production function was defined as:

$$Y=A*K^{(1-a)}*L^{a} \quad (1),$$

where: Y=the output, A= a constant of integration, K= the stock of fixed capital, L= the employment, a= the weight of the labour force remuneration, the gross value added. Initially, the asumption was that a= 0.75. The assumtion was validated by the estimations related to the USA manufacturing industry, during the 1899-1922 period, which permited to obtain the expression $Y=1.01*K^{0,25}*L^{0,75}$ (Frederenko, Kantorovici *et al*, 1979).

If we consider the weights of the two main production factors in the creation of the gross value added, the parameter A represents the index of the total factor productivity as defined in R. Solow (1957), *i. e*. the residium of the output change, which cannot be explained by the changes in the production factors' allocated quantities.

Because the sum of the weights of the two production factors is equal to unit, the initial form of the Cobb Douglas production function has constant scale returns. From an algebraical point of view, the parameter a is a particular case of the elasticity of the output with respect to labour force.

3. Algebraical properties of the estimated parameters of the Cobb-Douglas production function with constant returns to scale

If the constraints on output elasticity related to production factors are relaxed and the assumption that the constant returns to scale are maintained, the Cobb-Douglas production function defined by the expression

$$lnY = lnA_{1/k} + \alpha_1 \cdot lnK + (1 - \alpha_1) \cdot lnL \quad (2)$$

is equivalent to

$$lnP_L = lnA_{1/k} + \alpha_1 \cdot lnk \quad (3),$$

or

$$lnP_K = lnA_{1/k} + \beta_1 \cdot lnk \quad (4)$$

where:

ln Y= natural logarithm of the fixed base index of the output

ln K= natural logarithm of the fixed base index of the fixed capital stock

ln L= natural logarithm of the fixed base index of the employed population (labour force)

ln P_L= natural logarithm of the fixed base index of the productivity of labour

ln P_K= natural logarithm of the fixed base index of the productivity of fixed capital

ln k= natural logarithm of the fixed base index of the capital deeping

α_1= elasticity of productivity of labour with respect to capital deeping

β_1= elasticity of productivity of fixed capital with respect to capital deeping

$lnA_{1/k}$= constant of integration

Therefore, the Cobb-Douglas production function with constant returns to scale acts also as productivity of labour or as a productivity of fixed capital function. Hence, the function has a dual nature, because it is possible to reveal not only the behaviour of the productivity of labour, but also the beviour of the productivity of fixed capital during the analysed period.

If we use the OLS method for the estimation of parameters α_1, β_1 and ln $A_{1/k}$, we may write:

$$\alpha_1 = v_{PL/k} \cdot s_{PL/k} \quad (5)$$

$$lnA_{1/k} = lnP_{LR} \cdot (1 - s_{PL/k}) \quad (6)$$

$$v_{PL/k} = \frac{lnP_{LR}}{lnk_R} \quad (7)$$

$$s_{PL/k} = \frac{strcov(lnP_L;lnk)}{CV^2(lnk)} \quad (8)$$

Where:

ln P_{LR} = natural logarithm of the representative index of the labour productivity. (Representative index defined in F. M. Pavelescu (1986) as the geometrical mean of the fixed base indices of the analysed indicator)

lnk = natural logarithm of the representative index of the capital deeping

strcov (lnP_L;lnk) = structural component of the covariance of the natural logarithms of fixed base indices of labour productivity and the natural logaritms of fixed base indices of capital deeping

$$strcov(lnP_L; lnk) = \frac{cov(lnP_L;lnk)}{lnP_{LR} \cdot lnk_R} \quad (9)$$

CV (lnk) = variation coefficient of the logarithms of the fixed base indices of the capital deeping

F.M. Pavelescu (2017) demonstrates that the elasticity of labour productivity with respect to capital deeping (α_1) and the elasticity of productivity of fixed capital with respect to capital deeping (β_1) are linked by the relationship:

$$\alpha_1 = 1 + \beta_1. \quad (10)$$

Hence, it results:

$$\beta_1 = v_{PK/k} \cdot s_{PK/k} \quad (11),$$

where: $v_{PK/k}$ and $s_{PK/k}$ are similar to $v_{PL/k}$ and $s_{PL/k}$.

Formula (10) shows that the elasticity of productivity of labour with respect to capital deeping is essentially influenced by the feature of productivity of fixed capital dynamics. Depending on the value of β_1 we are able to appreciate the apparent efficiency of the replacement of the labour force with the fixed capital. If $\beta_1 > 0$, we may admit that the respective substitution is an efficient one[1]. If $\beta_1 < 0$, the efficiency of the production factor substitution is ambigous and we have to use other indicators in order to determine the appreciate the efficiency of the changes made in the technical structure of the analysed economic entity.

F.M: Pavelescu (2017) also shows that the residual factor ln $A_{1/k}$ obtained in case of the simple linear regression runned for estimation of the parameter α_1 is the same with the residual factor obtained in case of simple linear regresion runned for the estimation of the parameter β_1.

The residual factor of the simple linear regression (3) may be also written as:

$$lnA_{1/k} = \ln P_{LR} - \alpha_1 \cdot lnk_R \quad (12)$$

Hence, it is possible to reveal the residual character of ln$A_{1/k}$. In other words, we may consider the respective parameter as an estimation gap of the natural logarithm of representative productivity index.

4. Identification of the types of correlations of the estimated parameters of the Cobb-Douglas production function with constant returns to scale in the context of the growth of labour productivity

The estimated parameters of the Cobb-Douglas production function depend on the dynamics of the partial productivity of the considered production factors. In the long run, the growth of labour productivity acts as a fundamental economic law and represents one of the main conditions for economic development. The experience accumulated, especially during the industrialization period, revealed that both the productivity of labour and capital deeping tended to increase, while the productivity of fixed capital either increased or decreased. In this context, labour force acts as a substituted production factor, while fixed capital acts as a substituting production factor.

Hence, we are able to identify six cases of the correlation among the estimated parameters α_1, $v_{PL/k}$, $s_{PL/k}$, $lnA_{1/k}$, β_1, $v_{PK/k}$, $s_{PK/k}$ (Table 1).

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1 In fact, formula (10) is, in a way, similar to the relationship between the elasticity proper of the output with respect to the considered production factor (δ_1) and the elasticity of the productivity of the considered production factor with respect to respective production factor (ε_1). F. M. Pavelescu (2003) demonstrates that $\delta_1 = 1 + \varepsilon_1$.

Table 1 - Correlations of the estimated parameters of the Cobb-Douglas production function with constant returns to scale and their modelling factors in the context of the growth of both labour productivity and capital deeping

Case	$\boldsymbol{\alpha_1}$	$\boldsymbol{v_{PL/k}}$	$\boldsymbol{s_{PL/k}}$	$\boldsymbol{lnA_{1/k}}$	$\boldsymbol{\beta_1}$	$\boldsymbol{v_{PK/k}}$	$\boldsymbol{s_{PK/k}}$
1	>1	>1	<1	>0	>0	>0	<1
2	<1	>1	<1	>0	<0	>0	>1
3	>1	>1	>1	<0	>0	>0	>1
4	>1	<1	>1	<0	>0	<0	<1
5	<1	<1	>1	<0	<0	<0	<1
6	<1	<1	<1	>0	<0	<0	>1

The assumption of continous growth of the capital deeping was challenged by of the second transition to market economy xperience by of the Central and Eastern European countries and by the growing role of the service sector in the consolidated market economies. Due to the above-mentioned structural changes, the productivity of labour grew even in a context of the decrease of capital deeping. Hence, apparently, the labour force has replaced the fixed capital.

Consequently, both the elasticity of productivity of labour and elasticity of productivity of fixed capital with respect to capital deeping are negative. In order to avoid some impediments in the economic interpretation generated by negative elasticities, it is recommandable to use the inverse of the capital deeping (k_{inv}) as an explanatory variable. Hence, we consider the expresions:

$$lnP_L = lnA_{1/k} + \alpha_{1c} \cdot \ln k_{inv} \qquad (13),$$

and

$$lnP_K = lnA_{1/k} + \beta_{1c} \cdot \ln k_{inv} \qquad (14)$$

We notice that the intercept of the considered simple linear regressions (13) and (14) is not dependent on the use of lnk or lnk_{inv} and is the same with the intercept estimated in case of simple linear regressions (3) and (4). Also, we have $\alpha_{1c}= -\alpha_1$ and $\beta_{1c}= -\beta_1$

Therefore, we are able to identify three cases of the correlations of the estimated parameters α_{1c}, $lnA_{1/k}$ and β_{1c} and their modelling factors (Table 2).

Table 2 - Correlations between the estimated parameters of the Cobb-Douglas production function with constant returns to scale and their modelling factors in the context of the growth of labour productivity and the diminish of capital deeping

Cazul	β_{1c}	$v_{PK/kinv}$	$s_{PK/kinv}$	$lnA_{1/k}$	α_{1c}	$v_{PL/kinv}$	$s_{PK/kinv}$
1	>1	>1	<1	>0	>0	>0	<1
2	<1	>1	<1	>0	<0	>0	>1
3	>1	>1	>1	<0	>0	>0	>1

5. Quantification of the total factor productivity dynamics in the context of the estimation of the parameters of the Cobb-Douglas production function with constsnt returns to scale

Formula (12) shows some similarities with the computation formula of the natural logarithm of the representative index of total factor productivity ($lnPTF_R$):

$$lnPTF_R = \ln P_{LR} - w_K \cdot lnk_R \quad (15)$$

where: w_K= the weight of expenditures associated with fixed capital in the total costs related to considered production factors.

In literature, there is a broad consensus that a feasable assumption is to consider that the weight of fixed capital is one-third and the weight of labour is two-thirds (S. Ayar and C-J Dalgaard, 2005). The above-mentioned weights may be used when there are no reliable statistical data. Of course, when there are reliable statistical data, we may compute the weights of the two production factors by adopting different assumptions related to the equilibrium level of the respective weights in particular situations (E. Dobrescu, 2006).

Hence, in order to compute the representative index of total factor productivity by considering the estimated parameters of the Cobb-Douglas production fiunction with constant returns to scale, we have to use formula:

$$lnPTF_R = \ln A_{1/k} + (\alpha_1 - (\tfrac{1}{3})) \cdot lnk_R \quad (16),$$

equivalent to:

$$lnPTF_R = \ln A_{1/k} + (\beta_1 + (\tfrac{2}{3})) \cdot lnk_R \quad (17)$$

We note that the **representative index of total factor productivity is sensibly higher than the residual factor (intercept) estimated in case of the Cobb-Douglas production function with constant returns to scale, especially when the productivity of fixed capital tends to increase**.

If, during the analysed period there is a trend of decrease in capital deeping, and we have to use lnk_{inv}, the computation formulae of the representative index of total factor productivity are:

$$lnPTF_R = \ln A_{1/k} + (\alpha_{1c} + (\tfrac{1}{3})) \cdot lnk_{invR} \quad (18),$$

equivalent to

$$lnPTF_R = \ln A_{1/k} + (\beta_{1c} - (\tfrac{2}{3})) \cdot lnk_{invR} \quad (19)$$

We note that if we consider the formula (7), we may re-write the formula (16) as:

$$lnPTF_R = (v_{PL/k} - (\tfrac{1}{3})) \cdot lnk_R \quad (20),$$

equivalent to:

$$lnPTF_R = (v_{PK/k} + (\tfrac{2}{3})) \cdot lnk_R \quad (21)$$

The main advantage of the formulae (20) and (21) is that they permit to reveal the impact of the intensity of production factors substitution on the dynamics of total factor productivity.

It is important to note that in the context of the estimation of the parameters of the Cobb-Douglas production function with constant returns to scale we are able to compute the representative index of the total factor productivity. The above-mentioned index depends on the number of years of the analysed period and by the (concave or convex) trajectory of the dynamics of the considered indicator. Therefore, it is a bit difficult to make a comparison with the dynamics of the overall eficiency of production factor utilization which is observed during other periods or in the within of the other economic entities.

A solution to surpass the above-mentioned inpediments is the estimation of the rate of disembodied technical change.

6. Modelling factors of the estimated disembodied technical change proper related to productivity of labour

The concept of disembodied technical chage, initially proposed in Tinbergen (1942), consider that - in the context of an economy where the research-development activities are consolidated - the output tends to grow continously. Hence, the time acts like a production factor and determines the expansion of the output or of the partial productivities. Therefore, we may estimate the rate of disembodied tehnical change proper related to productivity of labour ($\gamma_{1/PL}$)[1], by running the simple linear regression

$$lnP_L = \ln A_{1/tPL} + \gamma_{1/PL} * t \qquad (22)$$

where: t= time factor

We notice that **the rate of disembodied technical change proper is, in our case, the elasticity of productivity of labour with respect to the time factor**.

Considering the algebraical properties of OLS, we obtain:

$$lnA_{1/tPL} = lnP_{LR} \cdot (1 - R(lnP_L; t) \cdot \frac{CV(lnP_L)}{CV(t)}) \qquad (23)$$

$$\gamma_{1PL} = \ln(1 + rrP_L) \cdot R(lnP_L; t) \cdot \frac{CV(lnP_L)}{CV(t)} \qquad (24),$$

where:

rrP_L= representative rate of productivity of labour[2]

$R(\ln P_L; t)$ = Pearson coefficient of correlation of the natural logarithms of fixed base indices of labour productivity and time factor.

CV(t) = coefficient of de variation in the time factor

We can demonstrate that

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1 We use the notion of rate of disembodied technical change proper. Therefore, if we use the OLS method, the estimated parmeters obtained in case of simple linear regressions may be considered as proper ones. The estimated parameters of the multiple linear regressions are influenced by collinearity, which is quantified by the coefficient of collinear refraction. For this reason the respective results may be considered as derived values of the estimated parameters.

2 The representative rate was firstly defined in F. M. Pavelescu (1986), having in view the quntification of the rate of disembodied technical change in case of a Cobb-Douglas production function with non-constant returns to scale. The respective rate of disembodied technical change was one related to the output dynamics.

$$CV(t)^2 = \frac{n-1}{3\cdot(n+1)}, \qquad (25)$$

where:

n= number of years of the analysed period

ln(1+rrP_L) is a weighted arithmetical mean of the natural logarithms of the yearly indices of labour productivity, namely:

$$\ln(1 + rrP_L) = \sum_{p=1}^{n} \frac{2\cdot(n+1-p)\cdot lnAIdxp_{PL}}{n\cdot(n+1)} \qquad (26),$$

where:

$lnAIdxp_{PL}$= natural logarithm of the yearly index of productivity of labour corresponding to the year p.

We notice that **the representative rate is associated with a "memory effect". This means that the yearly indices corresponding to the beggining of the analysed period have weights , which are sensible higher in comparison with those corresponding to the end of the analysed period.**

The representative rate can be also computed by the formula:

$$\ln(1 + rr_{PL}) = \gamma_{1PL} + \frac{2\cdot lnA_{1/tPL}}{(n+1)} \qquad (27)$$

The representative rate may be compared with the yearly average rate (rmP_L). In case of a strictly exponential growth in the labour productivity, the representative rate is equal to the yealy average rate. Intuitively, we may observe that, in case of a concave trajectory of growth of labour productivity, the representative rate is higher than the yearly average rate, while in a context of a convex trajectory of growth in labour productivity, the representative rate is lower than the yearly average rate.

If we consider the difference between $\ln(1 + rrP_L)$ and $\ln(1 + rmP_L)$, we can obtain the following computation formula :

$$\gamma_{1PL} = \ln(1 + rmP_L)\cdot (1 + strcov(lnAIdxp_{PL}; t)) \cdot R(lnP_L; t) \cdot \frac{CV(lnP_L)}{CV(t)} \quad (28)$$

where:

strcov ($lnAIdxp_{PL}$;t) = structural component of the covariance of natural logarithms of yearly indices of labour productivity and time factor

Formula (27) reveals the correlation of the rate of disembodied technical change with the yearly average rate of productivity of labour.

On the other hand, we can demonstrate that:

$$\gamma_{1PL} = \theta_1 + \gamma_{1Pk} \tag{29}$$

where:

θ_1= elasticity of capital deeping with respect to time factor

$\gamma_{1/PK}$= elasticity of productivity of fixed capital related to time factor (rate of disembodied technical change related to productivity of fixed capital

We may also write:

$$\theta_1 = \ln(1 + rmk)\cdot (1 + strcov(lnAIdxp_k; t)) \cdot R(lnk; t) \cdot \frac{CV(lnk)}{CV(t)} \tag{30}$$

and

$$\gamma_{1PK} = \ln(1 + rmP_K)\cdot (1 + strcov(lnAIdxp_{PK}; t)) \cdot R(lnP_K; t) \cdot \frac{CV(lnP_K)}{CV(t)} \tag{31}$$

where:

rmk and rmP_K are similar to rmP_L, strccov ($lnAIdxp_k$;t) and strccov ($lnAIdxp_{PK}$;t) are similar to strccov ($lnAIdxp_{PL}$;t), R(lnk;t) and R(lnP_K;t) are similar to R(lnP_L;t), while CV(lnk) and CV(lnP_K) are similar to CV(lnP_L).

7. The modelling factors of the estimated parameters of the Cobb-Douglas production function with constant returns to scale and disembodied technical change

We may include the rate of disembodied technical change in the Cobb-Douglas production function with constant returns to scale and obtain a more complex production, which is described by the formula:

$$lnP_L = lnA_2 + \alpha_2 \cdot lnk + \gamma_2 \cdot t \tag{32}$$

The estimated parameters of the above-mentioned production function are:

$$\alpha_2 = \alpha_1 \cdot T_{2k} \tag{33}$$

$$\gamma_2 = \gamma_1 \cdot T_{2t} \tag{34}$$

$$lnA_2 = lnP_{LR} \cdot (1 - (s_{PL/k} \cdot T_{2k} + s_{PL/t} \cdot T_{2t})) \qquad (35),$$

where:

T_{2k} = coefficient of collinear refraction related to capital deeping

T_{2t} = coefficient of collinear refraction related to time factor

$$T_{2k} = \frac{1-R(lnk;t)\cdot r}{1-R^2(lnk;t)} \tag{36}$$

$$T_{2t} = \frac{r-R(lnk;t)}{r\cdot(1-R^2(lnk;t))} \tag{37}$$

$$r = \frac{R(lnP_L;t)}{R(lnP_L;lnk)} \tag{38}$$

If both coefficients of collinear refraction are positive, we deal with a non-harmful collinearity. If one of the coefficients mentioned above is negative we face a harmful collinearity.

The size of the estimated parameter lnA_2 is also influenced by the collinearity. In case of perfect collinearity, we have:

$$lnA_2 = \frac{lnA_{1/k}+lnA_{1/t}}{2} \tag{39}$$

We mention that the identification of conditions of the occurence of harmful collinearity in a multiple linear regression with two explanatory variable and also the impact of collinearity on the size of the intercept are largely discussed in F.M. Pavelescu (2017).

If we estimate the Cobb-Douglas production function with constant returns to scale and disembodied technical change by considering the productivity of fixed capital as a dependent variable, we obtain the following multiple linear regression:

$$lnP_K = lnA_2 + \beta_2 \cdot lnk + \gamma_2 \cdot t \quad (40),$$

where:

$$\alpha_2 = 1 + \beta_2 \quad (41),$$

We note that lnA_2 și γ_2 take the same values as those estimated in case of the linear regression au valori identice cu cele obținute în cazul regresiei (32).

If both the productivity of labour and capital deeping tend to grow, we are able to identify 8 types of correlations between the estimated parameters α_1, α_2, β_1, β_2 and γ_2, depending on the feature of productivity of fixed capital dynamics and the occurence of the harmful collinearity.

A. **If the productivity of fixed capital tends to decrease we can identify 4 possible types of correlations**:

1) $\alpha_1 < 1 < \alpha_2$; $\gamma_2 < 0$; $\beta_1 < 0 < \beta_2$. Harmful collinearity is manifest in case of the disembodied technical change related to productivity of labour and also in case of the elasticity of productivity fixed capital with respect to capital deeping, because we have:

$$T_{2/kPL} > 1, T_{2/tPL} < 0 , T_{2/kPK} < 0, T_{2/tPK} > 1$$

2) $\alpha_1 < \alpha_2 < 1$; $\gamma_2 < 0$, $\beta_1 < \beta_2 < 1$. Harmful collinearity occurs only in case of the dissembodied technical change related to productivity of labour, because we have:

$$T_{2/kPL} > 1;\ \ T_{2/tPL} < 0; \qquad 0 < T_{2/kPK} < 1;\ \ 0 < T_{2/tPK} < 1$$

3) $0 < \alpha_2 < \alpha_1 < 1$; $\gamma_2 > 0$, $\beta_2 < \beta_1 < 0$. Harmful collinearity is manifest only in case of the elasticity of productivity fixed capital with respect to capital deeping, because we have:

$$0 < T_{2/kPL} < 1; \quad 0 < T_{2/tPL} < 1 ; \qquad T_{2/kPK} < 0; \quad T_{2/tPK} > 1$$

4) $\alpha_2 < 0 < \alpha_1 < 1$;$\gamma_2 > 0$, $\beta_2 < \beta_1 < 0$. Harmful collinearity occurs both in case of elasticity of labour productivity with respect to capital deeping and in case of the rate of disembodied technical change related to fixed capital productivity, because we have:

$$T_{2/kPL} < 0; \quad T_{2/tPL} > 1 ; \quad T_{2/kPK} > 1; \quad T_{2/tPK} < 0.$$

B. **If productivity of fixed capital grows we identify other 4 types of correlations**:

5) $1 < \alpha_1 < \alpha_2$; $\gamma_2 < 0$, $0 < \beta_1 < 0 < \beta_2$, Harmful collinearity occurs in case of the rate of disembodied technical change related both to labour productivity and to fixed capital productivity, because we have:

$$T_{2/kPL} > 1, \quad T_{2/tPL} < 0, \qquad T_{2/kPK} > 1, \quad T_{2/tPK} < 0 .$$

6) $1 < \alpha_2 < \alpha_1$; $0 < \gamma_2$, $0 < \beta_2 < \beta_1$, There is no harmful collinearity, because we have:

$$0 < T_{2/kPL} < 1, \quad 0 < T_{2/tPL} < 1, \quad 0 < T_{2/kPK} < 1, \quad 0 < T_{2/tPK} < 1$$

7) $\alpha_2 < 1 < \alpha_1$; $0 < \gamma_2$, $\beta_2 < 0 < \beta_1$ Harmful collinearity is manifest only in case of the elasticity of fixed capital productivitywith resperct to capital deeping, because we have:

$$0 < T_{2/kPL} < 1, \quad 0 < T_{2/tPL} < 1, \quad T_{2/kPK} < 0, \quad T_{2/tPK} > 1.$$

8) $1 < \alpha_2 < \alpha_1$; $0 < \gamma_2$, $\beta_2 < 0 < \beta_1$, Harmful collinearity occurs both in case of elasticity of labour productivity with respect to capital deeping and in case of elasticity of fixed capital productivity with respect tto capital deeping, because we have:

$$T_{2/kPL} < 0, T_{2/tPL} > 1, \ T_{2/kPK} < 0, T_{2/tPK} > 1.$$

If capital deeping tends to decrease, we hve to run the following multiple linear regressions:

$$lnP_K = lnA_2 + \beta_{2c} \cdot lnk_{inv} + \gamma_2 \cdot t \tag{42}$$

and

$$lnP_L = lnA_2 + \alpha_{2c} \cdot lnk_{inv} + \gamma_2 \cdot t \tag{43}$$

We notice that

$$\beta_{2c} = 1 + \alpha_{2c} \tag{44}$$

Analogously, we can identify the 8 types of correlations of the estimated parameters α_1, α_2, β_1, β_2 și γ_2 and the possible occurence of harmful collinearity.

8. Proposal for an extended estimation methodology of the parameters of Cobb-Douglas production function with constant returns to scale and disembodied technical change

The review of the algebraical properties of the OLS method provides arguments in favour of an extended estimation methodology for the parameters of the Cobb-Douglas production function with constant returns to scale and disembodied technical change. This way, we are able to provide a feasable quantification of the total factor productivity dynamics, to identify the features of the partial productivities dynamics and the correlations of the productivity of substituted production factor with the intensity of production factors substitution. The proposed methodology has five steps, namely:

1) **Estimation of the simple linear regression lnk=** $\boldsymbol{lnB_1 + \theta_1 \cdot t}$. Therefore, we are able to compute the representative index and rate of the capital deeping and also the Pearson coefficient of correlation R(lnk;t). We note that the estimated parameter θ_1 is a component of the rate of disembodied technical change related to productivity

of labour. Depending on the sign of θ_1, we choose the form of the estimation of the production function parameters.

2) **Estimation of the simple linear regressions** $lnP_L = lnA_{1/k} + \alpha_1 \cdot lnk$ **and** $lnP_K = lnA_{1/k} + \beta_1 \cdot lnk$ **or** $lnP_L = lnA_{1/k} + \alpha_{1c} \cdot lnk_{inv}$ **and** $lnP_K = lnA_{1/k} + \beta_{1c} \cdot lnk_{inv}$

Therefore, it is possible to reveal: a) the elasticity proper of the productivity of labour and productivity of fixed capital with respect to capital deeping; b) modelling factors of the above-mentioned elasticities; c) computation of the representative index of total factor productivity.

3) **Estimation of the simple linear regressions** $lnP_L = lnA_{1/tPL} + \gamma_{1PL} \cdot t$ și $lnP_K = lnA_{1/tPK} + \gamma_{1PK} \cdot t$ .So, we are able to identify the modelling factors of the rates of disembodied technical change proper related to labour productivity and to fixed capital productivity, respectively.

4) **Estimation of the multiple linear regressions** $lnP_L = lnA_{1/k} + \alpha_2 \cdot lnk + \gamma_2 \cdot t$ and $lnP_K = lnA_{1/k} + \beta_2 \cdot lnk + \gamma_2 \cdot t$ or $lnP_L = lnA_{1/k} + \alpha_{2c} \cdot lnk_{inv} + \gamma_2 \cdot t$ and $lnP_K = lnA_{1/k} + \beta_{2c} \cdot lnk_{inv} + \gamma_2 \cdot t$. So, we can determine the coefficients of collinear refraction and the collinearity types, which occur in the considered multiple linear regresions.

5) **Computation of the simple arithmetical mean of the intercepts** $lnA_{1/k}$ and $lnA_{1/t}$(($lnA_1)_{am}$). The comparison of ln A_2 with $(lnA_1)_{am}$ reveal the impact of departure from perfect collinearity on the estimated value of the intercept of the Cobb-Douglas production function with constant returns to scale and disembodied technical change.

9. A numerical example. Estimation of the parameters of the Cobb-Douglas production function with constant returns to scale and disembodied tehnical change in for the economy of Japan during the 1971-1986 period

Based on the data from *National Accounts Statistics. Main Aggregates and Detailed Tables*, New York, 1982, 1985, 1989 and *Economic Survey of Europe 1990-1991,* *Statistical Appendix*, ECE-UN 1991 for the 1972-1986 period and on interpolation of the data for the year 1971, we further practically use the proposed methodology for the estimation of the parameters of the Cobb-Douglas production function with constant returns to scale and disembodied technical change in case of the economy of Japan during the 1971-1986 period.

Step1. The simple linear regression related to natural logarithms of fixed base indices of capital deeping and time factor led to the following results:

lnk=0,2234+0,0540*t R(lnk;t) = 0,9765, R^2=0,9536, F=287,9633

(7,2581) (16,9696)

N.B. The Student test statistics are presented in brackets. R^2*= coefficient of determination,*

F= Fisher test statistics.

We notice that ln(1+rrk)= 0,0803, $s_{k/t}$= 0.6727 and ln(1+rmk)= 0,0630

Step 2. The simple linear regressions of labour productivity and fixed capital productivity, respectively, on the capital deeping are:

lnP_L=-0,0445+0,5299*lnk R(lnP_L;lnk) = 0,9828, R^2=0,9659, F=397,1543

(-2,3218) (19,9287)

lnP_K=-0,0445-0,4701*lnk R(lnP_K;lnk) = -0,9783, R^2=0,9515, F= 312.6649

(-2,3218) (-17,6823)

It results that $v_{PL/k}$= 0,4640, $s_{PL/k}$= 1,1420, $v_{Pk/k}$= -0,5360, $s_{PL/k}$= 0.8771.

The natural logarithms of representative indices of labour productivity, capital deeping and fixed capital productivity are: lnP_{LR}= 0,3167, lnk_R= 0,6825 și lnP_{kR}= -0,3658. Hence, it results that $lnPTF_R$= 0,0892.

Step 3. Estimation of the simple linear regression related to labour productivity and fixed capital productivity, respectively, and time factor led to the following results:

lnP_L=0,0644+0,0297*t R(lnP_L;t) = 0,9951, R^2=0,9902, F=1414,1960

(8,4457) (37,6058)

We notice that $\ln(1+rrP_L)$= 0,0373, $s_{PL/t}$= 0.7965 and $\ln(1+rmP_L)$= 0,0198

Hence, we may conclude that the productivity of labour grew on a concave trajectory.

$\ln P_K$=-0,0159-0,0243*t R($\ln P_K$;t) = -0,9158, R^2=0,8386, F=72,7607

(-5,7602) (-8,5299)

We notice that $\ln(1+rrP_K)$= -0,0430, $s_{PK/t}$= 0,5655 and $\ln(1+rmP_K)$= -0,0229

Based on the comparison of $\ln(1+rrP_K)$ with $\ln(1+rmP_K)$ we may conclude that the trajectory of the decrease of fixed capital productivity was a concave one.

We are also able to verify the assumption that $\gamma_{1PL} = \theta_1 + \gamma_{1PK}$, because γ_{1PL} =0,0297, θ_1= 0,0540 and γ_{1PK} = -0,0243

Step 4. Estimation of the parameters of the Cobb-Douglas production function with constant returns to scale and disembodied technical change allows us to obtain the following results:

If we consider the labour productivity as the dependent variable, we have:

$\ln P_L$= 0.0356+0.1289*lnk +0.0227*t R^2= 0.9928 F= 902,116

(2,4152) (2,1940) (6,9913)

The coefficients of collinear refraction are: $T_{2k/PL}$= 0,2432 and $T_{2t/PL}$= 0,7654.

If we consider the productivity of fixed capital as a dependent variable, we have:

$\ln P_K$= 0.0356-0,8711*lnk +0.0227*t R^2= 0.9910 F= 715,4017

(2,4152) (-14,8316) (6,9913)

The coefficients of collinear refraction are: $T_{2k/PK}$= 1,8530 and $T_{2t/PK}$= -0,9342

Step 5. The intercept of the multiple linear regression ($\ln A_2$) is equal to 0.0356. We note that $\ln A_{1/k}$=-0.0445 and $\ln A_{1/tPL}$=0.0644. Hence we have $\ln(A_1)_{am}$=0.0097

Based on the estimation results, we are able to conclude that, during the analysed period, capital deeping saw a sensible increase on a concave trajectory, the labour acted as the substituted production factor, while the fixed capital was the substituting production factor. We notice that the labour productivity sensibly grew in the context of a sensible decrease in fixed capital productivity. The trajectory of the dynamics of both the labour productivity and fixed capital productivity was a concave one. The intense production factors substitution have essentially determinated the fast growth in the total factors productivity. It is important to observe that the logarithm of the representative index of total factor productivity is sensibly higher than the intercept of the simple linear regression considering the labour productivity or the fixed capital productivity as the dependent variable. The intercept of the Cobb-Douglas production function with constant returns to scale is negative. In fact we deal with the case 6 presented in Table 1. The type of the labour productivity growth described by the parameters of the above-mentioned production function is representative for an economic development during the industrial era, where the substitution of the labour by the fixed capital was one of the most important consequences of the technical change.

The results of the in estimation of the Cobb Douglas production function with constant returns to scale and disembodied technical change show that there is a non-harmful collinearity from the point of view of the explanation of the dynamics of labour productivity and a there is a harmful collinearity from the point of view of explanation of dynamics of fixed capital productivity. The harmful collinearity occurs in case of the rate of disembodied technical change. This is a consequence of the fact that we deal with the situation 3, presented in the classification of the above-mentioned types of productivity of labour growth related to productivity of fixed capital and time factor. In fact, this is a consequence of a relatively stable increase of the productivity of labour and a relatively unstable decrease of the productivity of fixed capital.

The average of the intercepts of the two simple linear regression of the dynamics of productivity of labour is positive, due essentially to the intercept obtained in case of the linear regression related to labour productivity and time factor. The intercept computed in case of the Cobb-Douglas production function with constant returns to scale and disembodied technical change is higher than the above-mentioned average. Hence we are able to conclude that the departure from perfect collinearity favours the increase in the size of the intercept.

Conclusions

The numerical example presented above confirms the theretical assumptions and algebraical properties of the OLS method. This a proof that the estimated parameters of the Cobb-Douglas production function and the Cobb-Douglas production function with constant retruns to scale and disembodied technical change may included in several situations, which are very well defined. The definition of the respective situations help the modeller to provide a correct interpretation of the analysed dynamics of the output or of the production factor productivity. In this context, it offers a good opportunity for a feasable determination of the dynamics of total factor productivity. Also, in case of the Cobb-Douglas production function with constant returns to scale and disembodied techical change we are able to reveal the impact of collinearity on the estimated parameters.

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